2024-10-31 MA3F1 SAUL SCHLEIMER LECTURE 15

① COROLLARIES of NO RETRACT THM.

<u>TABLECLOTH THEOREM</u>: SUPPOSE $f : \mathbb{D}^2 \to \mathbb{D}^2$ HAS $f|S^1 = \mathrm{Id}_{S^1}$. THEN f IS SURJECTIVE.

<u>PERRON-FROBENIUS</u> $(n=3)$: SUPPOSE $A \in \mathrm{MAT}_{3,3}(\mathbb{R})$ HAS $a_{ij} > 0$ FOR ALL i,j. THEN A HAS AN EIGENVALUE $\lambda > 0$ AND A λ-EIGENVECTOR v WITH $v_i > 0$ FOR ALL i.

THE NEXT RESULT REQUIRES MORE WORK TO PROVE

<u>FUNDAMENTAL THM of ALGEBRA</u> SUPPOSE $Q \in \mathbb{C}[z]$ IS A NON-CONSTANT POLYNOMIAL. THEN THERE IS SOME $z_0 \in \mathbb{C}$ SO THAT $Q(z_0) = 0$. [THAT IS, Q HAS A ROOT.]

② <u>NULL HOMOTOPIES</u>:

SUPPOSE $f : X \to Y$ IS GIVEN. WE SAY f IS <u>NULL-HOMOTOPIC</u> IF $f \simeq e$ [A CONSTANT MAP]

<u>DEF</u>: A POINTED MAP $f : (X, x_0) \to (Y, y_0)$ IS <u>NULL HOMOTOPIC REL BASEPOINT</u> IF THERE IS A HOMOTOPY $F : X \times I \to Y$ WITH $f_0 = f$, $f_1 = e$ $\left[\begin{array}{l} e(x) = y_0 \text{ FOR} \\ \text{ALL } x \in X \end{array}\right]$, AND $F(x_0, t) = y_0$.

THAT IS $f \underset{x_0}{\simeq} e$.

<u>PROPOSITION</u>: FOR $X = S^1$ THE FIRST IMPLIES THE SECOND.

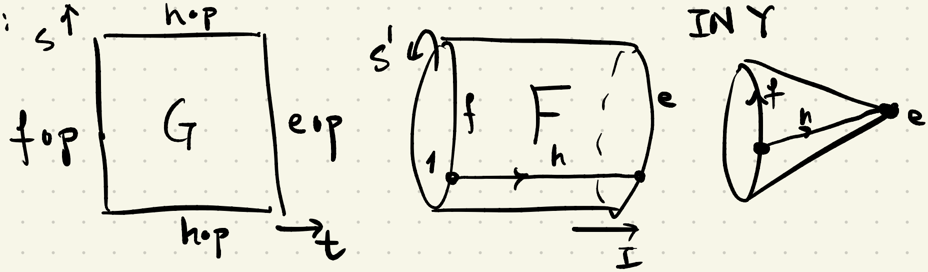
[THIS IS <u>NOT</u> THE CASE FOR GENERAL X ]

<u>PROOF</u>: SUPPOSE $F : S^1 \times I \to Y$ IS THE GIVEN HOMOTOPY. DEFINE $G : I \times I \to Y$ BY

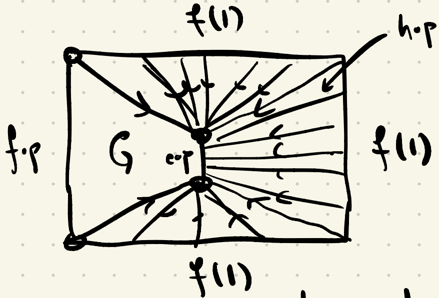
$$G(s,t) = F(p(s), t)$$

DEF $h : I \to Y$ BY $h(t) = F(1,t)$. FOR $p(s) = \exp(2\pi i s)$

PICTURE:



SO BUILD $H : I \times I \longrightarrow Y$ BY



FINALLY DEFINE

$K : S^1 \times I \longrightarrow Y$

$(z,t) \longmapsto \cancel{G}\,H\left(\frac{\log(z)}{2\pi i}, t\right)$

THIS IS THE DESIRED POINTED HOMOTOPY.

DEFINE $p_k : S^1 \longrightarrow S^1$ BY $p_k(z) = z^k$.

COROLLARY: $p_k \simeq p_\ell$ IFF $k = \ell$. $\square$

SO: p_k IS NULL HOMOTOPIC IFF $k = 0$.

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③ PROOF SKETCH of FUND. THM of ALGEBRA.

SUPPOSE, FOR A CONTRADICTION, THAT HAS NO ROOT. SO $Q(\mathbb{C}) \subset \mathbb{C} - \{0\}$. FOR $R \in \mathbb{R}_{>0}$ DEFINE

$$C_R = \{ z \in \mathbb{C} \mid |z| = R \}$$

SUPPOSE $Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, WITH $a_n \neq 0$. SINCE IT DOES NOT CHANGE THE ROOTS WE MAY DIVIDE AND SO ASSUME $a_n = 1$.

NOW PICK $R >> 0$ SO THAT $R^n >> \sum_{k=0}^{n-1} |a_k| R^k$

WITH THIS CHOICE of R

$$Q|C_R : C_R \longrightarrow \mathbb{C} - \{0\}$$

IS HOMOTOPIC TO

$$\begin{aligned} q: C_R &\longrightarrow \mathbb{C} - \{0\} \\ z &\longmapsto z^n \end{aligned}$$

IN FACT BY STRAIGHT-LINE HOMOTOPY.

SO: q IS HOMOTOPIC TO $p_n : S^1 \longrightarrow \mathbb{C} - \{0\}$

$$z \longmapsto z^n$$

SO $Q|C_R$ IS <u>NOT</u> NULL-HOMOTOPIC IN $\mathbb{C} - \{0\}$.

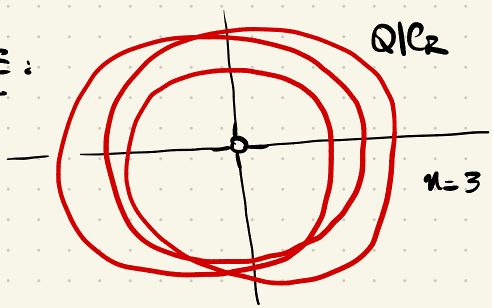
NOW CONSIDER

$$\left.\begin{aligned} F: C_R \times I &\longrightarrow \mathbb{C} - \{0\} \\ (z, t) &\longrightarrow Q(t \cdot z) \end{aligned}\right\}$$

THIS IS A NULL-HOMOTOPY of $Q|C_R \times \{0\}$.

THIS IS THE DESIRED CONTRADICTION $\square$

<u>PICTURE</u>:



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④ <u>EVEN / ODD</u>. SUPPOSE X, Y ARE SUBSETS of VECTOR SPACES. SUPPOSE X, Y ARE INVARIANT UNDER NEGATION.

<u>DEF</u>: SAY $f: X \to Y$ IS

$$\left.\begin{aligned} &\text{EVEN} \\ &\text{ODD} \end{aligned}\right\} \text{ if } \left\{\begin{aligned} f(-x) &= f(x) \\ f(-x) &= -f(x) \end{aligned}\right.$$

EXAMPLES: (1) $\sin : \mathbb{R}^1 \to \mathbb{R}^1$ IS ODD

(2) $\cos : \mathbb{R}^1 \to \mathbb{R}^1$ IS EVEN

(3) $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ IS ODD.

(4) $P_k : \left.\begin{array}{l} S^1 \longrightarrow S^1 \\ z \longmapsto z^k \end{array}\right\}$ IS $\left\{\begin{array}{c} \text{EVEN} \\ \text{ODD} \end{array}\right\}$ IF $k \equiv \left\{\begin{array}{c} 0 \\ 1 \end{array}\right\} \bmod 2$
