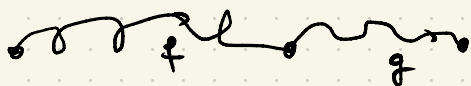


(1) PATHS: FOR $f, g: I \rightarrow X$ WITH $f(1) = g(0)$

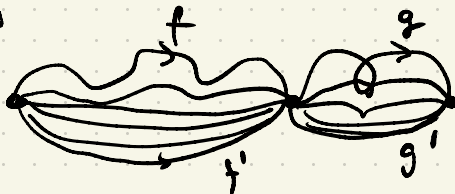
DEFINE $f * g: I \rightarrow X$ BY $(f * g)(t) = \begin{cases} f(2t) & t \leq 1/2 \\ g(2t-1) & t \geq 1/2 \end{cases}$



EXERCISE: IF $f \approx f'$, $g \approx g'$ AND $f(1) = g(0)$

THEN $f'(1) = g'(0)$ AND $f * g \approx f' * g'$

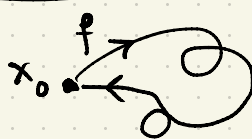
PICTURE:



(2) LOOPS: SUPPOSE X IS A SPACE, $x_0 \in X$ IS A BASEPOINT. CALL THE PAIR (X, x_0) A POINTED SPACE.

DEF A PATH $f: I \rightarrow X$ IS A LOOP BASED AT x_0

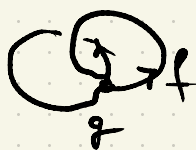
IF $f(0) = f(1) = x_0$. PICTURE



DEF:
LOOPS(X, x_0)
BELOW

NOTE: IF f, g ARE LOOPS AT x_0

THEN SO IS $f * g$. PICTURE:



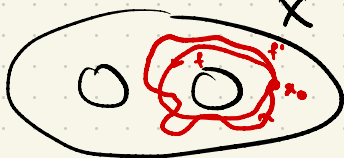
IN $\mathbb{R}^3$
SAY.

NOW DEFINE

$$[f] = \{ f' \text{ A LOOP AT } x_0 \mid f' \approx f \}$$

THE EQUIV. CLASS OF f .

PICTURE



CAN WIGGLE f TO f'
BUT CANNOT "JUMP
OVER HOLES".

③ THE FUNDAMENTAL GROUP

SUPPOSE (X, x_0) IS A POINTED SPACE.

DEFINE

$$\pi_1(X, x_0) = \left\{ [f] \mid \begin{array}{l} f: I \rightarrow X \\ \text{IS LOOP BASED} \\ \text{AT } x_0 \end{array} \right\}$$

DEFINE $[f] \cdot [g] = [f * g]$

EXERCISE: THIS IS WELL-DEFINED.

PROP 1.3: $(\pi_1(X, x_0), \cdot)$ IS A GROUP.

EXAMPLE: $\pi_1(\mathbb{R}^n, 0)$ IS THE TRIVIAL GROUP.

[NO HOLES!]

PROOF of 1.3: SUPPOSE f, g, h ARE ANY LOOPS AT x_0

SUPPOSE e IS THE CONSTANT LOOP AT x_0 .

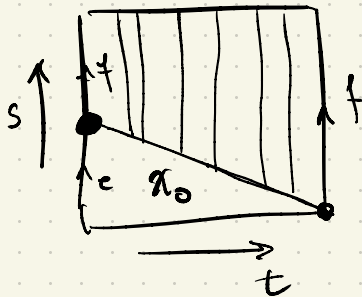
(i) CLOSURE: $f * g$ IS A LOOP AT x_0 .

THUS $[f] \cdot [g] \in \pi_1(X, x_0)$.

(ii) IDENTITY:

CLAIM: $[f] \cdot [e] = [e] \cdot [f] = [f]$.

PROOF:



DEFINE

$$F(s, t) = \begin{cases} x_0, & \text{if } 2s + t \leq 1 \\ f\left(\frac{2s + t - 1}{t + 1}\right), & \text{if } 1 \leq 2s + t \end{cases}$$

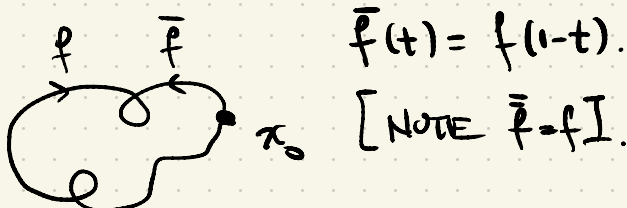
[FOR EXAMPLE]

THUS $e * f \cong f$. SIMILARLY, $f * e \cong f$.

SO $[e] \cdot [f] = [f][e] = [f]$. //

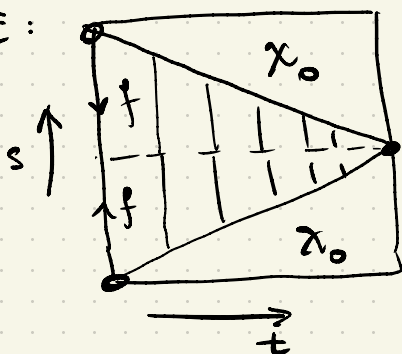
(iii) INVERSES. DEFINE $\bar{f}: I \rightarrow X$ BY

PICTURE:



CLAIM: $[f] \cdot [\bar{f}] = [\bar{f}][f] = [e]$.

PROOF:



IN COORDINATES $F(s,t) =$

$$= \begin{cases} x_0, & \text{if } 2s \leq t \\ f(2s-t), & \text{if } t \leq 2s \leq 1 \\ f(2-2s-t), & \text{if } 1 \leq 2s \leq 2-t \\ x_0, & \text{if } 2-t \leq 2s \end{cases}$$

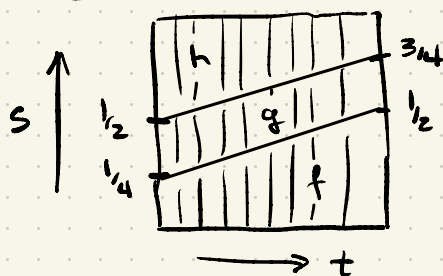
THUS $f * \bar{f} \cong e$. SIMILARLY $\bar{f} * f \cong e$.

SO $[f][\bar{f}] = [\bar{f}][f] = [e]$ //

(iii) ASSOCIATIVITY:

CLAIM: $([f][g])[h] = [f](g[h])$.

PROOF: WE PROVE $(f * g) * h \cong f * (g * h)$ VIA



THE PARAMETERISATION IS A BIT WORSE THAN THE CASE OF THE IDENTITY (ABOVE). EXERCISE! //

(*) had an incorrect sign in previous version.

DEF: SUPPOSE (X, x_0) IS PATH-CONNECTED. WE
CALL (X, x_0) SIMPLY-CONNECTED IF $\pi_1(X, x_0) \cong 1$
[TRIVIAL GROUP]

IMPORTANT EXAMPLE

$$X = S^1, x_0 = (1, 0).$$

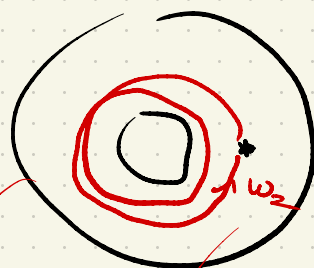
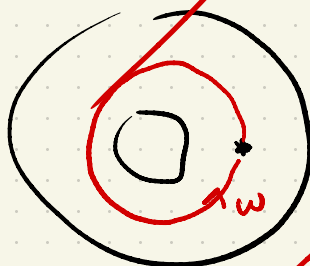
DEFINE $w_n(t) = (\cos(2\pi nt), \sin(2\pi nt))$.

NOTE $w_n: I \rightarrow S^1$ IS A LOOP.

DEF $e = w_0, w = w_1$

THEOREM 1.7: $\pi_1(S^1, x_0)$ IS ISOMORPHIC TO $\mathbb{Z}$
GEN. BY $[w]$.

PICTURES: WE THICKEN S^1 TO GET $A^2 = S^1 \times I \cong S^1$



SO w_n WINDS n TIMES AROUND THE ANNULUS.

EXERCISES: $w_1 * w_{-1} \cong w_0$

$$\bar{w}_1 = w_1$$

$$w_1 * w_1 \stackrel{(*)}{=} w_2 \quad (*) \text{ SPECIAL CASE!}$$

$$w_n * w_m \cong w_{n+m}$$

INTRODUCED NOTATION:

$$\text{LOOPS}(X, x_0) = \{ f: I \rightarrow X \mid \begin{matrix} f(0) = f(1) \\ = x_0 \end{matrix} \}$$

THE SET OF BASED LOOPS

SO $*$: $\text{LOOPS}(X, x_0) \times \text{LOOPS}(X, x_0) \longrightarrow \text{LOOPS}(X, x_0)$

IS BINARY OP: BUT (*) NO IDENTITY

(*) NO INVERSES

(*) NO ASSOCIATIVITY!

BUT HAS THESE THINGS "UP TO HOMOTOPY" SO MOD OUT

$\text{LOOPS}(X, x_0) \longrightarrow \pi_1(X, x_0)$ } AND WIN.
 $f \longmapsto [f]$